THE DECEMBER MEETING OF THE CHICAGO SECTION.

THE eighteenth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, December 29 and 30, 1905 Morning and afternoon sessions were held on each day, the first session opening on Friday at 10:30 o'clock. The following members were present:

Professor D. F. Campbell, Dr. E. L. Dodd, Professor J. F. Downey, Professor J. W. Glover, Professor A. G. Hall, Professor Thomas F. Holgate, Dr. L. C. Karpinski, Professor Kurt Laves, Mr. N. J. Lennes, Dr. A. C. Lunn, Professor E. H. Moore, Professor H. B. Newson, Dr. J. C. Morehead, Miss Ida M. Schottenfels, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. J. Townsend, Mr. R. E. Wilson, Professor J. W. Young.

Professor A. G. Hall was elected chairman of the meeting. This being the regular time for the election of officers, the Section directed the nominating committee to report an executive committee for the Section to consist of a chairman, a secretary, and one additional member; this committee to have charge of all business of the Section including the preparation of the programme.

The officers elected for the year 1906 were as follows: Chairman, Professor Alexander Ziwet; Secretary, Professor H. E. Slaught; additional member of the executive committee, Professor A. G. Hall.

By vote the Section expressed its appreciation of the services of Professor Thomas F. Holgate, the retiring secretary, who had served the Section since its organization.

The executive committee, together with Professors Townsend and Skinner, were constituted a committee to arrange at an early meeting for a full half day conference of the Section with the deans and professors of engineering in technical schools, on the teaching of mathematics to engineering students.

The executive committee was requested to arrange so that the Section may take up the study of questions relating to applied mathematics, to the teaching and history of mathematics, and to the establishment of closer relations with colleagues in engineering and other professional subjects.

On the evening of Friday a dinner was held in the Hotel Del Prado, at which nineteen persons were present.

The following papers were read at this meeting:

(1) Dr. H. L. COAR: "Functions of three real variables."

(2) Dr. E. L. DODD: "A theorem concerning implicit functions, and its geometric aspect."

(3) Professor KURT LAVES: "A dynamical interpretation of an integral of Jacobi's partial differential equation for the problem of a solid body rotating about a fixed point."

(4) Professor J. W. YOUNG: "On discontinuous groups defined by the normal curve of the fourth order in a space of four dimensions."

(5) Professor E. H. MOORE: "On the theory of systems of integral equations of the second kind" (preliminary communication).

(6) Professor J. W. GLOVER: "The teaching of actuarial theory in universities and colleges."

(7) Professor H. B. NEWSON: "Complex number systems and continuous groups" (preliminary communication).

(8) Dr. A. C. LUNN: "Sets of postulates for the trigonometric functions."

(9) Dr. J. C. MOREHEAD: "Numbers of the forms $2^k q \pm 1$."

(10) Mr. N. J. LENNES: "Curves in non-metrical analysis situs."

(11) Mr. N. J. LENNES: "A fundamental existence theorem in the calculus of variations."

(12) Mr. N. J. LENNES: "Note on the Heine-Borel theorem."

(13) Dr. A. E. YOUNG: "On certain isothermic surfaces."

(14) Professor J. B. SHAW: "Canonical forms of hypercomplex numbers" (preliminary communication).

(15) Professor JACOB WESTLUND: "Primitive roots of an ideal in an algebraic number field."

(16) Professor L. E. DICKSON: "On quadratic, hermitian, and bilinear forms."

(17) Professor L. E. DICKSON: "Expressions for the elements of a determinant in terms of the minors of a given order. Generalization of a theorem due to Studnička."

(18) Professor G. A. MILLER: "The groups of order p^m which contain exactly p cyclic subgroups of order p^a."

Dr. Coar was introduced by Professor Townsend. In the absence of the authors, the papers by Dr. Young, Professor Shaw, Professor Westlund, Professor Dickson, and Professor Miller were read by title.

Professor H. B. Newson introduced for general discussion the question of "The distribution of mathematical subjects during the first two years of the college course." This discussion was participated in generally by members of the Section and others present. The main question was the best possible arrangement of the work for these two years on the assumption that the student enters with algebra through the progressions and with plane and solid geometry. Professor Newson stated the problem before the state universities of the middle west, where in most places college and engineering students are taught in the same classes. The basis of the discussion was the new programme at the University of Kansas, and the opinion prevailed that college and engineering students should be given the same courses.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Dr. Coar spoke on functions of three real variables which are completely defined and single-valued for all values of the variables. In the case of such functions three kinds of limits exist: (1) the triple limit $\lim_{x=x_0}\lim_{y=y_0}\lim_{z=z_0} f(x, y, z)$; (2) the double simultaneous limit $\lim_{x=x_0}\lim_{y=y_0, z=z_0} f(x, y, z)$; (3) the simultaneous limit in three variables $\lim_{x=x_0,\ y=y_0\ z=z_0} f(x, y, z)$. Necessary and sufficient conditions for the existence of each of these were given. The conditions for the interchange of the order of sequence of these limits, of which there are six cases, were obtained, and questions of the continuity and discontinuity of the limiting functions were discussed. The function may converge uniformly to the limiting function in two ways, either with respect to one variable or with respect to two variables taken together. The existence of either of these cases involves that of the other. The adaptability of the ideas of limits to functions defined by a series of functions of two real variables was shown, as was the bearing of this on the problem of term by term integration in the case of simultaneous integrals.

2. Dr. Dodd pointed out that it is intuitionally evident that a continuous surface, passing through a point (a, b) of the (x, y)-

plane, will cut the plane in a continuous line through (a, b), provided the surface inclines in the same general direction near (a, b). This suggests the following theorem regarding implicit functions, a theorem which may be proved analytically:

If $f(x, y)$ is a continuous function of x and y, and a univariant function of y near (a, b), where f vanishes, then there will exist one and only one continuous function $y = \phi(x)$ such that $\phi(a) = b$ and $f[x, \phi(x)] \equiv 0$ when x is near a.

The hypothesis does not require that $\partial f/\partial y \neq 0$, nor even that this derivative should exist. The theorem can be generalized.

3. After Poisson had applied Lagrange's method of variation of constants to the problem in question, it was Richelot who, in 1850, obtained the same final differential equations of perturbed motion by applying to it the Hamilton-Jacobi method. The difficult task of finding a complete solution of Jacobi's partial differential equation necessitated very laborious transformations. By the introduction of a certain spherical triangle Radau brought about a simplification of the method. In Professor Laves's paper the attempt is made to accentuate the geometric aspects of the problem by giving to the integral under investigation a dynamical interpretation which seems to have escaped the attention of those who have worked on the problem.

4. Fricke has defined arithmetically an extensive class of discontinuous groups of linear fractional substitutions of the form

$$\zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta} \qquad (\alpha\delta - \beta\gamma = 1)$$

as the reproducing groups of ternary quadratic forms. Professor Young's paper aims at a generalization of Fricke's procedure. The conic represented by the ternary quadratic form is replaced by a normal curve C_n of order n in a space S_n of n dimensions. This curve is transformed into itself by a group of ∞^3 collineations in S_n, each collineation subjecting the points of C_n, which depend rationally on a parameter ζ, to a substitution of the form $\zeta' = (\alpha\zeta + \beta)/(\gamma\zeta + \delta)$. If the coefficients in the collineations are restricted to be integers with determinant unity, the collineation groups on the z_i, the point coördinates in S_n, and the holoed-

rically isomorphic group on the parameter ζ will be discontinuous. Thus every C_n defines a discontinuous ζ-group.

Professor Young first shows that the ζ-groups defined as above by two normal C_n's that can be transformed into each other by a rational collineation (*i. e.*, one with rational coefficients) are commensurable, *i. e.*, they have a subgroup of finite index in common; for this reason it is natural to confine oneself to the groups defined by C_n's obtained from a canonical form by irrational collineations. The author then restricts himself to the case $n = 4$, and in particular to C_4's obtained from the canonical form: $z_1 = \zeta^1$, $z_2 = \zeta^3$, $z_3 = \zeta^2$, $z_4 = \zeta$, $z_5 = 1$ by transformations of the form:

$$z_1' = \sqrt{p}z_1 + \sqrt{q}z_5, \qquad z_2' = \sqrt{m}z_2 + \sqrt{n}z_4, \qquad z_3' = \sqrt{r}z_3,$$
$$z_4' = \sqrt{m}z_2 - \sqrt{n}z_4, \qquad z_5' = \sqrt{p}z_1 - \sqrt{q}z_5,$$

where p, q, m, n, r are positive integers. No essential restriction is involved in supposing that none of these integers contains a square factor and that p is prime to each of the remaining four. It is then found that the ζ-group $[p, q, m, n, r]$ will exist (*i. e.*, consist of more than the identical substitution) only if $r = p$, and further that this group is essentially the same as the reproducing group of the ternary quadratic form $Pz_1^2 - Qz_2^2 - Rz_3^2$, where P is the highest common factor of q and m, Q the highest common factor of m and n, and R the highest common factor of q and n.

5. Denoting (1) by R a range of values of arguments or indices s, t; (2) by F a field of ordinary complex numbers; (3) by Γ_{1t} a class or category of single-valued functions f_t of s over R, with functional values belonging to F; (4) by Γ_{2st} a category of single-valued functions K_{st} of (s, t) over (R, R), with functional values belonging to F; (5) by $\overset{\frown}{t}$ an operation applicable to functions of Γ_{1t} and serving to eliminate t from such functions, — Professor Moore studies postulates involving the symbols $(R, F, \Gamma_1, \Gamma_2, \overset{\frown}{\ })$ in such a way as to yield a theory of the linear equation of the second kind,

$$g_s = f_s + \overset{\frown}{K_{st}^{t}} f_t,$$

holding as s ranges over R, a theory which suitably specialized becomes a theory of the system of linear integral equations of the second kind,

$$g_i(x) = f_i(x) + \sum_{j=1}^{m} \int_a^b K_{ij}(x, y) f_j(y) dy \qquad \left(\begin{smallmatrix} i=1, \cdots, m \\ x=a, \cdots, b \end{smallmatrix}\right).$$

8. Dr. Lunn's paper gave several sets of independent postulates which, with the accompanying definitions, sufficed to determine the sine, cosine, and tangent, as real functions of a real variable.

9. Dr. Morehead's paper is supplementary to his paper read at the April, 1905, meeting of the Society, on numbers of the forms $2^k q \pm 1$ and Fermat's numbers. For the purpose of determining the primes and the prime factors of non-primes of the forms $2^k q \pm 1$, two positive functions, $q_k(p)$, $q_k'(p)$, of the odd prime p were defined. These functions possess the properties: $2^k q + 1 \equiv 0 \bmod p$, for $q = np + q_k$. $2^k q' - 1 \equiv 0 \bmod p$, for $q' = np + q_k'(n = 0, 1, 2, \cdots)$ and conversely. Thus for $q_{2^n} = 1$ we have $2^{2^n} \cdot 1 + 1 \equiv 0 \bmod p$, *i. e.*, p is a factor of the Fermat number $F_n = 2^{2^n} + 1$. From the various properties of q_k, q_k' is developed a means of computing q_{2^n}, which involves in general less labor than the usual method of computing successively the residues with respect to p of $2^{2^1}, 2^{2^2}, 2^{2^3}, \cdots 2^{2^n}$ in order to determine whether or not p is a factor of F_n.

The second part of the paper gives several theorems on the quadratic characters of q_k, q_k' with respect to the corresponding prime p, and shows how some, and often all, of the quadratic residues and non-residues of p can be written down at once.

10. In the first part of this paper Mr. Lennes gives fresh proofs of classical theorems on the separation of the plane by simple finite and infinite polygons and by a simply closed Jordan curve. The frontier of a region is considered and a set of conditions are specified under which such frontier is a Jordan curve. The definition of a continuous curve is at once translated into non-metrical language and the treatment is non-metrical throughout. In the second part of the paper several definitions of a continuous curve are given, one of which is developed as follows:

A set of points is connected if in every pair of complementary subsets at least one subset contains a limit point of points in the other set.

A set $\{P'''\}$ is said to separate a connected set $\{P\}$ into two sets $\{P'\}$ and $\{P''\}$ if every connected set lying in $\{P\}$ and containing points of both $\{P'\}$ and $\{P''\}$ also contains at least one point of $\{P'''\}$.

A closed, bounded, connected set of points containing the points A and B is a simple arc connecting A and B if every point of the set, except A and B, separates it into two sets, one of which contains A and the other B. Two simple arcs connecting A and B and having no point in common except A and B constitute a Jordan curve.

This definition of continuous curves applies equally well to curves in an n-dimensional manifold as to curves in the plane.

11. In this paper Mr. Lennes considers a set of arcs of Jordan curves connecting a fixed pair of points A and B. The set of arcs lies in a closed region R, having a connected interior. The expression "entirely open region" is used to denote a set of points, no point of which is a limit point of points not in the set. An entirely open region of the region R is a set of points of the region R no part of which contains a limit point of points of R which are not of the set.

In the region R is considered the set $[C_{AB}]$ of all arcs connecting A and B which satisfies the following condition:

For every point P of the region R and for every triangle t of which P is an interior point there exists a triangle t' of which P is an interior point such that no arc of the set $[C_{AB}]$ contains a point on t between two of its points on t'. (This may be regarded as a uniform continuity assumption on the set $[C_{AB}]$.)

Associated with each arc C_{AB} of the set $[C_{AB}]$ is a positive number $M(C_{AB})$ forming a set of numbers $[M(C_{AB})]$ corresponding to the set of arcs $[C_{AB}]$. The set of numbers $[M(C_{AB})]$ is assumed to satisfy the following condition: For every given arc C_{AB} of the set $[C_{AB}]$ and for every given positive number ϵ, there exists an entirely open region R' of R, such that for every arc C''_{AB} of $[C_{AB}]$ and of R', $M(C'_{AB}) - M(C''_{AB}) < \epsilon$. With these assumptions it is proved that there exists at least one arc $C^{(m)}_{AB}$, of the set $[C_{AB}]$ such that $M(C^{(m)}_{AB})$ is the greatest lower bound of the set $[M(C_{AB})]$.

12. In this note Mr. Lennes formulates the Heine-Borel theorem on non-metrical hypotheses, as follows:

If in a three-dimensional continuous manifold every point of a closed set $[P]$ is an interior point (fails to be a limit point of points not of the set $[R]$) of at least one set of points $[R]$ of a set of sets $[[R]]$, then there exists a finite subset $[R]_1$,

$[R]_2, \cdots, [R]_n$ of the set $[[R]]$, such that every point of $[P]$ is an interior point of at least one of the sets $[R]_1, \cdots, [R]_n$.

The following corollary is a useful form for metrical analysis:

If every point of $[P]$ lies within at least one of the sets of $[[R]]$ then there exists a positive number δ such that every cube whose edge is δ and whose center is a point of the set $[P]$ lies within at least one of the sets of $[[R]]$.

13. In a paper presented at the sixteenth meeting of the Chicago Section (December, 1905), Dr. Young gave the result of a certain investigation leading to the consideration of some new isothermic surfaces, which, when referred to lines of curvature, have their linear element in the form $ds^2 = (u+v)^n(du^2/U + dv^2/V)$, where n is a constant and U and V are functions of u and v, respectively.

In the present paper he discusses the problem of determining all surfaces corresponding to the linear element above when referred to lines of curvature. Two distinct cases have to be considered, the one where $n = \pm 2$, and the other where n has any other value. In the latter case Dr. Young shows first that the most general forms of the functions U and V are Laurent's series in u and v, respectively. A further discussion of this case shows that if the functions U and V are assumed to be analytic, then $n = \pm 1$ and U and V take very simple forms. When $n = 1$, one form of the functions U and V gives the sphere, while another form gives the other quadric surfaces. Likewise, when $n = -1$, a certain minimal surface corresponds to the first form of the functions U and V and certain other surfaces peculiarly related to the other quadrics correspond to the second.

The surfaces corresponding to the case $n = -2$ are the cyclides of Dupin and certain other envelopes of spheres, whereas the surfaces corresponding to $n = +2$ are the new surfaces which were considered in the previous paper.

This discussion is of particular interest in that it gives a good example of the comparative ease with which positive results may be obtained by working from the Gauss and Codazzi equations. All the quadratures are performed and the cartesian coördinates of all the new surfaces are given as functions of u and v.

14. Professor Shaw's paper develops the theorem that any

number in any linear associative algebra can be written in the form

$$\phi = \sum_{i,j} A_{i,j}^{(s \cdots \mu_i)} \eta_{i,j} \qquad (i, j = 1, \cdots, \rho),$$

where $\eta_{i,j}$ are quadrate vids, subject, that is, to the law $\eta_{ij}\eta_{jk} = \eta_{ik}$; $\eta_{ij}\eta_{kl} = 0$ if $j \neq k$. The quantities $A_{i,j}^{(s \cdots \mu_i)}$ are polynomials in a nilpotent ν, such that the power $\nu^{\mu_1+1} = 0$; the coefficients of the powers of ν are from any field, scalar, abstract, etc. The powers of ν in $A^{(s \cdots \mu_i)}$ start with the sth and run consecutively to the μ_ith. The number s is such that $s = 0$ if $\mu_i \leqq \mu_j$, $s = \mu_i - \mu_j + 1$ if $\mu_i > \mu_j$. The numbers μ_i $(i = 1, \cdots, \rho)$ are integers assigned beforehand and are such that $\mu_1 \geqq \mu_2 \geqq \mu_3 \cdots \geqq \mu_\rho$.

The consequences of this theorem are developed partially; in particular, for Peirce algebras, that is, non-quaternion algebras with no skew units. Such algebras are generated by ρ linearly independent, but not necessarily productly independent units. When their polynomials are known the algebra is determined. The persistent coefficients of these polynomials under linear transformation determine the different algebras of the same type, in terms of their "natural units." This paper is a continuation of "Theory of linear associative algebra" and "Nilpotent algebras" by the author (*Transactions*, Volume 4, 1903).

15. Professor Westlund's paper is a continuation and extension of a former paper. It is shown that in an arbitrary algebraic number field the only ideals which have primitive roots are of the form P^n, $Q_1^n Q_2 \cdots Q_i$, and $Q_1 Q_2 \cdots Q_i P^n$, where p, the rational prime divisible by P, is odd, and $Q_1, Q_2, \cdots$ are the prime ideal factors of 2. A method is also given for determining the primitive roots in each case.

16. The first paper by Professor Dickson treats of the reduction to canonical types of quadratic and hermitian forms in a general field. Application is made to quadratic and bilinear forms invariant under a given substitution S and their normalization by substitutions commutative with S. The latter problem is a generalization to an arbitrary finite or infinite field of the recent memoir by Jordan in the *Journal de Mathématiques* on the case of a finite field of prime order. The paper will appear in the *Transactions*, April, 1906.

17. The second paper by Professor Dickson gives a practical method of expressing the elements of a determinant D of order n as rational functions of the minors M_m of order m and the mth root of a rational function of the M_m. If from $m-1$ rows of D we build $n-m$ minors $M_{m-1}^{(i)}$ such that the elements of M' are chosen from any columns other than a fixed column C_{j_1}, the elements of M'' from any columns other than C_{j_1} and C_{j_2}, etc., we can express $M'M''\cdots M^{(n-m)}D$ as a determinant of order $n-m+1$ whose elements are minors M_m of D. For the case $M'=M''=\cdots=M^{(n-m)}$ the theorem is due to Studnicka. The latter case enables us to derive the minors M_{m-1} from the M_m, then the M_{m-2} from the M_{m-1}, and finally the elements from the M_2. This symmetric solution is valueless since it employs a complicated array of radicals of various degrees. The other solutions given involve a single irrationality, viz., the mth root of a rational function of the M_m. The paper appeared in the *American Mathematical Monthly*, December, 1905.

18. Professor Miller considers the possible types of groups G of order p^m (p any prime) which contain just p cyclic subgroups of a given order p^α ($\alpha<m$). He finds that every group G of this kind must contain a cyclic subgroup of order p^{m-2}. If p is restricted to be odd and $m>4$, a stronger theorem is obtained as follows: When p is odd and $m>4$, G contains exactly p cyclic subgroups of every order from p^2 to p^{m-1}. This theorem is also true when $m=3$, and when $m=4$ and $p>3$. As all the groups of order p^m which contain a cyclic subgroup of order p^{m-2} are known, these results give a complete determination of all the groups of order p^m, which contain exactly p cyclic subgroups of the same order.

THOMAS F. HOLGATE,
Secretary of the Section.

EVANSTON, ILLINOIS,
January 16, 1906.

THE GROUPS CONTAINING THIRTEEN OPERATORS OF ORDER TWO.

BY PROFESSOR G. A. MILLER

(Read before the American Mathematical Society, February 24, 1906.)

IT is known that a group of order 2^m cannot contain exactly thirteen operators of order 2, but that there are other groups which have this property.* Let G represent such a group. We shall first consider the case where the operators of order 2 in G form a single set of conjugates. In this case G transforms these operators according to a transitive substitution group of degree 13. The subgroup of G which corresponds to identity in this transitive substitution group will be represented by H.

If H were of even order, it would contain an operator of order 2 which would be commutative with every operator of this order contained in G. From this it would follow that each of these operators would be commutative with all of them. This is impossible since they cannot all be contained in a group of order 2^m. Hence *H is of odd order.*

The substitution group which is simply isomorphic with G/H must therefore contain substitutions of order 2. Since the group of order $2h$, h being the order of H, which corresponds to the group generated by such a substitution contains at least one operator of order 2, the substitution group of degree 13 contains exactly 13 substitutions of order 2; hence it involves only one subgroup of order 13 and its order is a divisor of 156. Moreover, *all the operators of H are commutative with every operator of order 2 contained in G.*

The product of two operators of order 2 contained in G corresponds to an operator of order 13 in the isomorphic group. Since this product is transformed into its inverse by each of its factors,† and since the operators of H are commutative with each of these factors, it follows that this product is of order 13. That is, *any two operators of order 2 contained in G generate the dihedral rotation group of order 26.* As each of the operators of this group is commutative with every operator of H, it fol-

* BULLETIN, vol. 12 (1905), p. 74.
† BULLETIN, vol. 7 (1901), p. 424.

lows that G contains the direct product of a group of odd order and the dihedral rotation group of order 26 as an invariant subgroup. The quotient group of G with respect to this invariant subgroup is contained in the cyclic group of order 6.

The order of G cannot be divisible by eight when all the operators of order 2 are conjugate. If this order is divisible by four the subgroups of order 4 are cyclic. Hence it follows that each of the Sylow subgroups of G whose order is a power of two contains only one operator of order 2 when the operators of this order in G form a single conjugate set. Conversely all the operators of order 2 must be conjugate under G whenever a Sylow subgroup of order 2^m contains only one operator of this order. Hence it follows that in the groups which remain to be considered each Sylow subgroup of order 2^m contains more than one operator of order two.

§ 1. *Groups Containing a Set of two Conjugate Operators of Order* 2.

Let s_1, s_2 be two operators of order 2 contained in G, such that s_1 is transformed into s_2 by exactly half the operators of G. As every operator which transforms s_1 into s_2 must also transform s_2 into s_1, G contains operators whose orders are powers of 2 which transform s_1 into s_2. Hence the Sylow subgroups of order 2^m contained in G are non-abelian and their orders must be divisible by 8. As the product $s_1 s_2$ is invariant under G, it is included in every Sylow subgroup of order 2^m. The Sylow subgroups which contain s_1 must therefore also contain s_2. That is, *every Sylow subgroup of order* 2^m *includes the four group generated by* s_1, s_2.

Every Sylow subgroup of order 2^m includes other operators of order 2. Let s_3 represent such an operator. We shall first prove that s_3 cannot be commutative with s_1 and s_2. If this were the case, a Sylow subgroup S would contain at least seven commutative operators of order 2. It could not contain exactly seven operators of order 2, since 10 is not divisible by 4. It could not contain eleven operators of this order;* for there would be some other Sylow subgroup S_1 which would contain

* This is a special case of the theorem: if a Sylow subgroup of any group G contains more than one subgroup of order p without including all the operators of order p in G, then G contains at least p^2 subgroups of order p which are not in a given Sylow subgroup.

at least two operators of order 2 which are not in S. One such operator would transform S into S_2 having at least two operators of order 2 which are not in S. As S_2 could not contain this transforming operator, it follows that S_1 would contain at least two operators which are neither in S nor in S_2. Since S_2 would contain at least two other operators of order 2 which are not in S, it is clear that S could not contain more than nine operators of order 2.

If S contained nine operators of order 2 and if s_3 were also commutative with s_1 and s_2, the operators of order 2 which are not in the group generated by s_1, s_2, s_3 could not be commutative with s_1. Hence s_1, s_2, s_3 would generate an invariant subgroup of S. As an operator of order 2 could not transform this subgroup into itself without being commutative with at least four of its operators, S could not contain exactly nine operators of order 2. Hence, as stated above, s_3 cannot be commutative with s_1 and s_2.

Since s_3 transforms s_1 and s_2, it, together with these two operators, generates the octic group. We shall now consider the case where S contains no operator of order 2 besides those of this octic group. As all the conjugates of S have three operators in common, there are just five distinct octic subgroups in these conjugates. These subgroups are transformed by all the operators of G according to a transitive group T of degree 5. Since T contains a complete set of five conjugate operators of order 2, it can involve only one subgroup of order 5. The operators of G which correspond to identity of T constitute an invariant subgroup H of G. The operators of H which are commutative with s_3 constitute a subgroup of half the order of H. This subgroup H_1 includes s_1, s_2.

As H_1 contains only one operator of order 2, its Sylow subgroups of order 2^m either are cyclic or they contain cyclic subgroups of half their order while the remaining operators are of order 4. They must therefore contain an odd number of cyclic subgroups of order 4 if they contain one such subgroup. If the order of H_1 were divisible by 4, H would contain an operator of order 4 which would be transformed into its inverse either by s_1 or by s_3. Hence either H would contain more than three operators of order 2, or a divisor of G corresponding to an operator of order 2 in T would contain more than two such operators. As this is impossible, *H contains only one subgroup of order 4, viz., the one generated by s_1, s_2, and the order of H is not divisible by 8.*

Two operators of order 2 in G which correspond to two distinct operators of order 2 in T have for their product an operator whose order is divisible by 5 but by no other odd prime, since s_3 is commutative with all the operators of H_1. The order of this product could not be divisible by 4 since H does not include any operator of this order. Moreover, the two given factors may be so selected as to make the order of this product either 5 or 10. In the latter case they generate the dihedral rotation group of order 20. This group contains all the operators of order 2 in G except s_1 and s_2. These two operators are commutative with the operators of its cyclic subgroup of order 10, but they transform the remaining operators into themselves multiplied by $s_1 s_2$. Hence *the operators of order* 2 *in* G *generate the group of order* 40 *which contains* 13 *operators of order* 2.

As the order of H_1 is not divisible by 4 and as H_1 contains only one operator of order 2, it is the direct product of a group of odd order and $s_1 s_2$. This group of odd order is clearly invariant under G. In fact, H is the direct product of this group and the group generated by s_1, s_2. Hence G contains the direct product of a group of odd order and the group of order 40 which contains exactly thirteen operators of order 2. If G contained any other operators, it would contain this direct product as an invariant subgroup of half its order, and the remaining operators would correspond to operators of order 4 in T. The square of such an operator would therefore transform s_1 into s_2, since s_3 has this property. This is clearly impossible as s_1 has only two conjugates under G.

We have now proved that G is the direct product of a group of odd order and the group of order 40 which contains just thirteen operators of order 2, whenever s_3 transforms s_1 into s_2 and a Sylow subgroup contains no operators of order 2 besides those contained in the octic group generated by s_1, s_2, s_3. Moreover, the direct product of *any* group of odd order and this group of order 40 possesses the properties in question. It remains to consider the case when S contains more than one octic subgroup involving s_1, s_2. Each of these octic subgroups is transformed into itself by every operator of order 2 contained in S. There could not be more than three such octic subgroups involving s_1, s_2, since S cannot involve eleven operators of order 2. If there were three, one would be invariant under S. The operators of this subgroup would transform each of the others

into itself. Hence each would be transformed into itself by the other two, and the three would generate a group whose commutator subgroup would be 1, $s_1 s_2$. As such a group would contain more than nine operators of order 2, S cannot contain more than two octic subgroups involving s_1, s_2.

As one octic group would transform one other octic group into itself, it would transform two of the five contained in G into themselves. Hence G would have to transform the five octic subgroups according to a transitive group of degree 5 containing a transposition. This is clearly impossible. Hence *G contains a set of two conjugate operators of order 2 only when it is the direct product of the group of order 40 which contains thirteen operators of order 2 and some group of odd order.*

§2. *Proof that there is no Group in which the Number of Operators of Order 2 in a Complete Set of Conjugates is Three, Four or Five.*

If there were a set of three operators of order 2 which formed a complete set of conjugates under G, they would be transformed according to a group of degree 3 by all the operators of G. The subgroup H which would correspond to the identity in this group of degree 3 could not involve these three conjugates; for, if they were all contained in a four group, H could not contain the thirteen operators of order 2 since $13 \not\equiv 3 \bmod 4$. Neither could H contain just seven operators of this order, since they would be included in an invariant subgroup of order 8 and hence G would have to involve more than thirteen operators of order 2. Since the number of operators of order 2 which are in G without being also in H is divisible by 6, no other numbers require consideration. If the three given operators generated a group of order 8, this would be invariant, and hence this case is impossible. If the three conjugate operators in question were not in H, there would be just one more operator of order 2 in a divisor corresponding to a substitution of order 2 than in H. This is again impossible because $13 \not\equiv 3 \bmod 4$. Hence there cannot be a complete set of three conjugate operators of order 2 under G.

If G contained a complete set of four conjugate operators of order 2, at least two of these would be in a four group. If three of them were in this group, they would all be commutative and hence would generate an invariant subgroup of order 8. As

this is impossible, the four group which contained two of these four conjugates would not contain any other. It is easy to see that this four group would be transformed into itself by the other conjugates and that the only case which requires further consideration is when these four conjugates generate an invariant octic group, and when S contains nine operators of order 2. No two Sylow subgroups could have only five common operators of order 2, since one of the remaining operators of order 2 in one of these Sylow subgroups could not transform the other into itself and hence would transform it into one having additional operators of order 2. As this is impossible, any two Sylow subgroups would have at least seven common operators of order 2.

The eight operators of order 2 which are not in the octic subgroup generated by the four conjugate operators in question would therefore be divisible into four pairs such that each pair with this octic group would generate a subgroup involving just seven operators of order 2. As the number of subgroups involving nine operators of order 2 would be odd, and as all would be conjugate under G, this is impossible. The impossibility of constructing such a group G can also be proved by observing that the nine operators of order 2 in S would generate a group of order 32 which would have a commutator subgroup of order 2 and contain the given octic group as invariant subgroup. As such a group cannot be constructed, it is proved that G could not contain a complete set of four conjugate operators of order 2.

If G contained a complete set of five conjugate operators of order 2, it would transform them according to a transitive substitution group of degree 5. Let H represent the subgroup of G corresponding to identity in this transitive substitution group. As H cannot involve the five conjugate operators of order 2, it contains just three operators of this order. This is impossible, since each of these three operators would be commutative with each one of the five conjugate operators of order 2, and hence each divisor corresponding to a substitution of order 2 would contain just four operators of order 2.

§ 3. *Groups in which there is a Complete Set of Six Conjugates of Order* 2.

If G contains a complete set of six conjugate operators of order 2 it must transform them according to a transitive sub-

stitution group T of degree 6. The conjugate operators of order 2 cannot be contained in the subgroup H which corresponds to identity of T, since G cannot contain an invariant subgroup of order 8 which involves no operator of order 4. If these six conjugates correspond to the same substitution of T, H would contain five operators of order 2. The remaining two operators could not correspond to this substitution of T. Hence they would correspond to another invariant substitution. This is impossible since T cannot contain two invariant substitutions of order 2. As every non-invariant substitution of order 2 has at least three conjugates under T, the six conjugate operators in question correspond either to three or to six substitutions of T.

In the latter case H would be of odd order; for, if it were of even order the operators of order 2 which would correspond to two commutative conjugate substitutions of degree < 6 could not be commutative since their product could not be of order 2. That is, there would not be a complete set of six conjugates of order 2 in G. If H were of odd order, only one operator of G would correspond to each of the six conjugates in question, and at least one operator of order 2 would correspond to every substitution of order 2 in T. It follows directly from the properties of the T's which have six conjugates of order 2 that this is impossible, hence the six conjugate operators of order 2 in question correspond to three conjugate substitutions whose degree is < 6, and H involves only one operator of order 2. If these three substitutions of T were commutative, the corresponding operators of order 2 would also be commutative, or else they would generate a group whose commutator subgroup is of order 2.

The former of these two alternatives is clearly impossible. In the latter, the six conjugates in question would generate a subgroup of order 16, and a Sylow subgroup of order 2^m would involve at least 11 operators of order 2. As this is impossible, it follows that the three conjugate substitutions of T which correspond to the six conjugates in question must generate a group of order 6. These conjugates must therefore generate the dihedral rotation group of order 12, and hence this is an invariant subgroup of G, and T contains an invariant subgroup of order 3. From this it follows that T is one of the two groups of order 12.

We shall now prove that the order of H cannot be divisible

by 4. If it were divisible by 4, a Sylow subgroup of H would contain an odd number of cyclic subgroups of order 4. Hence one of them would be invariant under S. If an operator of order 2 not included among the six conjugates in question transformed the generator of this invariant subgroup into its third power, it would correspond to the invariant substitution of T and together with the given generator would generate the octic group. The two operators of order 2 which would be in the same divisor but not in this octic group would transform this octic group into itself. Hence this octic group would be invariant under the group generated by all the operators of order 2 in G. As the operators of G which would correspond to the invariant substitution of T without being in this octic group could not transform the operator of order 4 in this octic group into its inverse they would be commutative with it, and hence this divisor would contain operators of order 4. Such operators would be transformed into their inverse by some operator belonging to the six conjugates in question. As this would lead to operators of order 2 in a divisor corresponding to another substitution of T, it is impossible.

If an operator of order 2 corresponding to the invariant substitution of order 2 in T were commutative with the given operator of order 4, it would transform an operator of order 4 corresponding to one of the six conjugates in question into its inverse. As this is clearly impossible, it is proved that H could not involve an operator of order 4 in case an operator of order 2 in G corresponds to an invariant substitution in T. We shall now prove that the order of H is not divisible by 4 when each of the operators of order 2 in T corresponds to a non-invariant substitution in T. If H contained operators of order 4, such operators would correspond to each non-invariant substitution of order 2 in T. From this it follows directly that there would be operators of order 2 in the divisor corresponding to the invariant substitution of T. As this is impossible, it is proved that *the order of H is not divisible by* 4.

If the operators of order 2 which are not in the given set of six conjugates correspond to non-invariant substitutions of T, two operators of order 2 can be found such that their product is of order 12, since no operator of order 2 corresponds to the invariant substitution in T. Hence the operators of order 2 in G generate the dihedral rotation group of order 24, and G is the direct product of this group and the largest subgroup of

odd order contained in H. Moreover, every direct product of this dihedral rotation group and an arbitrary group of odd order contains just thirteen operators of order 2 which have the property in question.

If six operators of order 2 correspond to the invariant substitution of order 2 in T, they generate the dihedral rotation group of order 12, and H contains an invariant subgroup of order 3. As the largest subgroup of odd order in H transforms these six operators according to a group of odd order which has two transitive constituents of degree 3, it must contain an invariant subgroup of one-third its order which does not include the operator of order 3 in the said dihedral rotation group of order 12. Hence H is the direct product of a subgroup of order 2, a subgroup of order 3, and some group of odd order. Moreover, G contains the direct product of the dihedral rotation group of order 12, generated by the six operators of order 2 which are transformed according to T, and the invariant subgroup of order 3 in H.

The six operators of order 2 which correspond to the invariant substitution of T transform this direct product of order 36 into itself and generate with it the group of order 72 which contains exactly 13 operators of order 2 and is generated by these operators. This subgroup of order 72 is invariant in G and has only identity in common with the given group of odd order in H. Hence G is the direct product of some group of odd order and this group of order 72 whenever some of its operators of order 2 correspond to the invariant substitution of T, and every such direct product satisfies the condition in question. The result of this section may be expressed as follows: *Whenever G contains six operators of order 2 which constitute a complete set of conjugates, its thirteen operators of order 2 generate either a group of order 24 or a group of order 72 and G is the direct product of one of these groups and some group of odd order.* Every such direct product contains exactly six conjugates of order 2, and involves 13 operators of this order.

§ 4. *Proof that there is no Additional Group in which the Number of Operators of Order 2 in a Complete Set of Conjugates is Seven, Eight, Nine, Ten, or Eleven.*

If there were a complete set of either seven, eight or nine conjugates of order 2, the remaining operators of this order could not be invariant, since G cannot contain the group o order 8 which contains 7 operators of order 2 as an invariant

subgroup. These remaining operators could not occur in sets of conjugates, as these sets have been considered above. If G contained three invariant operators of order 2, while the remaining 10 formed a single set of conjugates, the three invariant operators would generate the four group. As any one of the remaining operators of order 2 and this four group would generate the group of order 8 which has 7 operators of order 2, the remaining operators could be arranged in distinct sets of four operators, which is evidently impossible.

If G contained two invariant operators, their product would also be invariant and hence there could not be a complete set of eleven conjugates. It only remains therefore to consider the case where G contains a single invariant operator of order 2 while the remaining 12 constitute a complete set of conjugates. It will be found that there are groups which come under this case. Hence there are groups containing exactly 13 operators of order 2 in which these operators form sets of conjugates containing any of the following numbers of operators: 13; 1, 2, 10; 1, 6, 6; 1, 12.

§ 5. *Groups in which there is a Complete Set of Twelve Conjugates of Order* 2.

Since such a G contains an invariant operator of order 2, it contains six conjugate four groups and transforms them according to a transitive substitution group T of degree 6. If the operators H of G which correspond to the identity of T included more than one operator of order 2, these six subgroups of order 4 would generate a group of order 2^m involving thirteen operators of order 2. As this is impossible, T contains a complete set of either three or six conjugate substitutions of order 2 and of degree < 6, corresponding to the operators of order 2 in G.

In the former case these three conjugate substitutions of T cannot be commutative, since all the operators of order 2 in G cannot be contained in a Sylow subgroup. Hence T is the group of order 12 which contains seven substitutions of order 2. The four operators of G which correspond to the same substitution in T generate the octic group. As the operators of order 2 in the other divisions transform this octic group into its conjugates, these three conjugates have the cyclic subgroup of order 4 in H in common. Hence this cyclic subgroup is invariant under G. From this it follows directly that H can-

not contain any operator of order 8; for if H contained such an operator, its square would generate the only invariant cyclic subgroup of order 4, but this square could not be in the given octic groups.

We shall now prove that the Sylow subgroups of H could not include the quaternion group. If this group occurred in H, a Sylow subgroup S of G would be of order 32 and would include two non-cyclic subgroups of order 16 which would involve only one operator of order 2. As it would contain another subgroup of order 16 involving a cyclic subgroup of order 8, it would contain just three cyclic subgroups of order 8. This is impossible since the number of such subgroups in a group of order 2^m is even.* The order of H is therefore divisible by 4 but not by 8. As all of the operators of odd order in H transform each operator of order 2 in G into itself but some of its operators of even order do not have this property, the operators of odd order generate a subgroup invariant under G, and H is the direct product of this subgroup and its cyclic subgroup of order 4.

Since the product of two non-invariant operators of order 2 in G is commutative with every operator of H, these two operators may be so chosen that their product is of order 12 and hence *the operators of order* 2 *in* G *must generate the dihedral rotation group of order* 24 *whenever four of the operators of* G *correspond to the same substitution in* T. As this invariant subgroup has only identity in common with the largest group of odd order contained in H, G includes the direct product of these two groups and its order is twice the order of this direct product.

If an operator of order 4 corresponds to the invariant substitution of T, all the operators of this divisor whose order is a power of two must be of order 4. Such an operator transforms each of the factors of the given direct product into itself. It transforms the subgroup of order 24 according to an operator of order 2, while it may be either commutative with the other factor or transform it according to an operator of order 2. In the former case G is the direct product of a group of odd order and the group of order 48 which involves just thirteen operators of order 2 and in which all the operators of order 4 are commutative with the two operators of order 3. This

* *Transactions Amer. Math. Society*, vol. 6 (1905), p. 59.

group of order 48 may also be defined by the fact that it contains the direct product of the quaternion group and the group of order 3, and is generated by this direct product and an operator of order 2 which transforms the operators of order 3 into their inverses and the quaternion group into a contragredient isomorphism with itself. In the latter of the two cases mentioned above G may be obtained by establishing an isomorphism between this group and a group whose order is twice an odd number.

When an operator of order 8 corresponds to the invariant substitution in T, all the operators of this divisor whose order is a power of two are of order 8. The remarks of the preceding paragraph apply directly to this case, with the exception that the given group of order 48 is replaced by the one which is generated by the cyclic group of order 24 and an operator which transforms each operator of this cyclic group into its eleventh power. Hence *there are two groups of order* 48 *which contain exactly* 13 *operators of order* 2, *twelve being conjugate, and transform their six four groups involving these operators according to just three distinct substitutions.* All the other possible groups which have this property can be obtained by dimidiating one of these two groups and a group whose order is twice an odd number and all such dimidiations give rise to groups having the required property.

We shall now consider the case where these six four groups correspond to six distinct substitutions of T. As these substitutions form a complete set of conjugates and are of degree less than six, T is either the positive group of order 24 or the group of order 48. If it were the latter, it would contain an invariant subgroup of order 8 times the order of H. As this subgroup would contain only one operator of order 2, it would include a Sylow subgroup having this property. The order of this S could not be less than sixteen. Hence it would have operators of order 8. This is impossible. In fact it is at once evident that such an S could not have an $(\alpha, 1)$ isomorphism with a group involving no operator of order 4 unless the order of this group is less than eight. The groups in question must, therefore, transform the six conjugate four groups according to the positive group of order 24, which is simply isomorphic with the symmetric group of degree 4.

The order of H is not divisible by four, since the operators which correspond to the three conjugate substitutions in T

must be all of a higher order than those of H. This follows also from the fact that the quaternion group is the only group of order 2^m involving only one operator of order 2 which has an operator of odd order in its group of isomorphisms. Hence H is the direct product of a group of order 2 and some group of odd order. Every operator of H is commutative with each of the operators of the group generated by all the operators of order 2 in G. Two of these operators corresponding to two commutative substitutions in T generate the octic group. Hence all the operators of H are commutative with the operators of order 4 corresponding to the three conjugate substitutions of T. That is, *G contains an invariant quaternion group and this subgroup includes all its operators of order* 4.

Two operators of order 2 can be so selected that their product is of order 3. This product and the given quaternion group generate a group of order 24 which is transformed into itself by any one of the operators of order 2. Hence *the thirteen operators of order* 2 *generate a group of order* 48, *and G is the direct product of this group and a group of odd order.* Moreover, any such direct product satisfies the given conditions. The given group is completely defined by the facts that it is of order 48 and is generated by thirteen operators of order 2. The main results of this section are as follows: If a group contains just thirteen operators of order 2 of which twelve are conjugate, these operators generate either the dihedral rotation group of order 24 or the group of order 48 which may be represented as a transitive group of degree 8 and contains the quaternion group invariantly. In the latter case G is the direct product of this group of order 48 and some group of odd order. In the former case G contains one of two groups of order 48 and may be constructed by dimidiating this group and a group whose order is twice an odd number.

If a group contains exactly thirteen operators of order 2 these operators must, therefore, generate one of the following five groups: The dihedral rotation group of order 24, the dihedral rotation group of order 26; the group of order 40 which contains thirteen operators of order 2; the group of order 48 which can be represented as a transitive group of degree 8 having four but not two systems of imprimitivity; or the group of order 72 which is generated by the direct product of the dihedral rotation group of order 12, an operator of order 3, and an operator of order 2 which transforms this operator of

order 3 into its inverse and the non-invariant operators of order 2 in the dihedral rotation group of order 12 into themselves multiplied by the invariant operator of order 2. In case of the groups of order 40, 48 and 72 all the possible groups are obtained by forming the direct product of these groups and some group of odd order.

STANFORD UNIVERSITY,
November, 1905.

TYPES OF SERIAL ORDER.

*The Continuum as a Type of Order: An Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers.** By EDWARD V. HUNTINGTON. Cambridge, Mass., The Publication Office of Harvard University, 1905. 4to. 63 pp. Price, 50 cents.

THE *Annals of Mathematics* has for some time followed the plan of printing articles expository of subjects which are little known or not easily accessible in the English language. Reprints of these articles are then placed on sale with the double and laudable purpose of making the circulation of the article wider than it would otherwise be and of helping solve the difficult problem of financing a mathematical journal.

The plan can hardly fail to succeed if all the articles are as clear in style and as just in the balance between generality and detail as is that of Professor Huntington. In point of readableness, we are inclined to think that the only other exposition of subjects connected with the foundations of mathematics which can be compared with Huntington's is that (in French) of L. Couturat.

The principal contents of the paper are the ordinal theory of integers, rational numbers, and the continuum, together with an appendix on the transfinite numbers of Cantor. It is intended for non-mathematical readers as well as for mathematicians, and therefore presupposes very little in the way of detailed knowledge, though of course it requires for complete comprehension a considerable maturity in abstract reasoning. We have noticed only one error of any consequence. It is

* Reprinted from the *Annals of Mathematics*, second series, vol. 6, No. 4 (July, 1905), and vol. 7, No. 1 (October, 1905).

stated in theorem 3, section 62, that "if a series is dense, it will also be dense in itself." That this is incorrect may be shown by an example (involving Cantor's order type Ω) similar to one given on page 169, volume 5, of the *Transactions*.

On page 184 appears an expression of opinion which is repeated in one form or another in several other places: "It is not possible to give purely ordinal definitions for the sums and products of the elements of such a series. All that we could do in this direction would be to define the sums and products of some particular dense series, say the series of the rational numbers in the usual order, by the use of some extraordinal properties peculiar to that series; then since all series of the type η are ordinarily similar, the definitions set up in the standard series could be transferred to any other series of the same type by a one-to-one correspondence. This method would be wholly inadequate, however, since the ordinal correspondence could be set up in an infinite number of ways." The reviewer is unable to see why it is incorrect to fix attention on a single correspondence and to base the definition on that one. The fact that there exist an infinitude of other correspondences giving similar definitions is a theorem which has no particular bearing on the validity of the method pursued in regard to the particular one with which we work. On the other hand, we do not wish to controvert the author's opinion that, from a didactic point of view, it is better to introduce such fundamental concepts as addition and multiplication at the outset of the theory.

In this connection it seems proper to refer to a logical distinction recently stated by Huntington. The reference to it here, however, must not suggest that Huntington allows such logical subtleties to interfere with the clearness of his elementary exposition. A mathematical science is said to be determined categorically by a set of postulates or conditions if it can be proved that any two classes of objects satisfying the conditions are capable of a one-to-one correspondence preserving the relations described in the postulates. For example, the Cantor definition of the linear continuum in terms of ordinal relations is such that any two sets of objects satisfying the conditions imposed (for example, the points of a parabola and the points of a straight line) are capable of a one-to-one correspondence preserving order. Huntington proposes to distinguish among

the ways in which a set of conditions may be categorical.* In the case just mentioned the one-to-one correspondence can be set up in an infinitude of ways, whereas if the continuum is defined in terms of postulates about addition and multiplication as well as order, the correspondence can occur in only one way. This is in part because there now are specified in any set of elements satisfying the postulates two singular elements, zero and unity, which must correspond to the zero and unity elements of any other set which satisfies the postulates.

The distinction suggested by Huntington can be carried a step further by counting parameters. Thus in the case of the linear continuum, if the postulates are stated in terms of order and addition alone there is only one singular element, zero, and the unity element may be chosen arbitrarily. Thus we have ∞^1 correspondences. If the zero is also allowed to be arbitrary but a relation of equality of segments (such as the additive operation gives by the equation $b - a = b' - a'$) be retained, then we have ∞^2 correspondences. If order alone is used in the definition then Cantor's method of showing the correspondence gives ∞^∞ correspondences. That is, there exists no n such that ∞^n will describe the possibilities of choice. We seem to have arrived at a criterion by which, given two systems of axioms, one may determine which system is, as it were, the more categorical.

In plane geometry there is an interesting question which can be decided in the same way. Hilbert's Festschrift, in addition to its axioms of connection and order and its archimedean axiom, makes use of assumptions about congruence of segments and angles, a relation which it has long been known may be introduced by definition. This definition involves the choice of an arbitrary elliptic involution on the line at infinity (*i. e.*, of two conjugate imaginary points, the circular points) and thus may be made in ∞^2 ways. The question, therefore, arises whether congruence, which as an undefined symbol in the presence of order and continuity axioms is clearly redundant, does not after all make the system of axioms more categorical?

In setting up the one-to-one correspondence between any two classes satisfying the axioms of plane geometry a pair of coördinate axes for an analytic geometry is selected in each class.

* Cf. page 159 of the paper under review and also *Transactions Amer. Math. Society*, vol. 6 (1905), p. 41.

If congruence axioms are present, a given line a of one class may correspond to any line a' of the ∞^2 lines in the other class. A given point A of a may correspond to any one of the ∞^1 points of a'. A line perpendicular to a at A, however, must correspond to a line perpendicular to a' at A'. The whole correspondence is therefore fixed when a unit of length $A'B'$ is chosen on a' to correspond to AB on a. There are thus ∞^4 ways in which the correspondence between two planes defined by order and congruence axioms may be set up. If order axioms alone are used, it is clear that the number of correspondences must be the same as the number of collineations of a cartesian plane into itself leaving the line at infinity invariant, and this count is ∞^6. A system of axioms stated in terms of order and congruence combined is therefore more categorical than one in terms of order alone.

OSWALD VEBLEN.

COLLEGE ALGEBRA.

A College Algebra. By HENRY BURCHARD FINE. Ginn and Company, 1905. viii + 595 pp.

AMONG all the text-books of college algebra, this book by Professor Fine, of Princeton, is distinguished by its broad grasp of the subject as a branch of mathematical science and by its entire freedom from the misleading statements and positive blunders which are all too common in our current text-books. What measure of success it will obtain in the class-room it would be hard to prophesy; my own impression is that on account of the close logical interdependence of its various parts the book will be found less available for those classes which wish to take up only a few detached chapters on separate topics than for those classes which can go through the whole subject in a systematic manner. However this may be, it is certainly a book which every teacher of mathematics, whether he happens to have a class in algebra or not, should own and read. In fact, if a mathematical library were to contain only one English text-book in algebra (besides the two-volume treatise of Chrystal) I should unhesitatingly recommend the present book.

The book is divided into two parts: "a preliminary part devoted to the number system of algebra, and a principal part devoted to algebra itself."

The first part (about one-seventh of the whole) contains a systematic treatment of the rational, irrational, and imaginary numbers, in the light of the researches of Dedekind and Cantor; these researches have been known on the continent for over thirty years, but are here for the first time made accessible in a satisfactory manner in an English text-book.* If there is anyone who feels that such discussions are unimportant, on the ground that the old-fashioned treatment of irrationals and imaginaries in our current algebras is well enough as it is, let him ask any freshman class in mathematics at the opening of the college year what meanings the words "irrational number" and "imaginary number" convey to them. If his experience is anything like my own, he will find that not a single member of the class has any clear-cut notion of these concepts. The replies obtained will be simply variations on the following, which are entirely typical of those I have received: "An irrational number is one which has no exact value;" "An imaginary number is one which has no value at all and does not really exist." Interesting objects, indeed! And yet these are the mystical entities which the school boy is taught to add and subtract, to multiply and divide, as if they were ordinary numbers in the arithmetical sense. No wonder that an occasional bright pupil will protest that "you cannot extract the square root of a number which hasn't got any square root," and assure you that "to put a radical sign around a minus quantity always seemed like trying to squeeze money out of a man who didn't have any"! Such are the fruits of the present half-hearted treatment of these subjects in the ordinary text-books; any fresh and rigorous discussion, like that of Professor Fine's, should therefore be cordially welcomed by every lover of common sense and common honesty in teaching.

How much of this preliminary part will be available for the class-room, every teacher must decide for himself; the college student at any rate should be made to grasp at least the main principles which underlie the work, namely: (1) that the number system is merely a system of signs which we invent for our own convenience in handling certain practical problems (like the problems of counting and measuring); (2) that the "operations" on these numbers which are almost inevitably suggested by the

*Since this book was published, another excellent account of the Dedekind-Cantor theory has appeared in Pierpont's Theory of functions of real variables.

exigencies of those problems are essentially defined by the fundamental laws which govern them (*e. g.*, the commutative, associative, and distributive laws for addition and multiplication); and (3) that all the propositions of algebra are logically deducible from these fundamental laws without further reference to the nature of the numbers involved.

It will be seen that from this point of view the definitions of the numbers themselves are not important; the fundamental laws of operation are all that we need to know. In other words, *any* system which obeys the laws of rational algebra is a system of rational numbers; *any* system which obeys the laws of real algebra is a system of real numbers; etc.* To fix our ideas, however, and also to show that the laws are not inconsistent, it is desirable to exhibit some concrete example of such a system. The principal example which Professor Fine selects for this purpose (to speak only of the real numbers) is the one proposed by Dedekind and Cantor, as already mentioned; in this system, the rational numbers are pictured as pairs of integers and the real numbers as "separations" in the series of rationals (or as "regular sequences" of rationals). Later on another example is introduced, paralleling the first, in which the real numbers are pictured as points on a line (or better, as distances along a line). For the purposes of an elementary course it would have been preferable, I think, if the author had given the principal place to the second or more geometrical of these two parallel methods of representation. In all the formal processes of real algebra, it makes no difference whether the letters a, b, x, y, etc., stand for "separations in a series of pairs of integers" or for "distances along a line"; and surely the latter interpretation is more accessible to an elementary student, and much more useful in practice. The only advantage which the Dedekind-Cantor example possesses lies in the fact that it enables us to establish the consistency of the laws of real algebra on purely logical grounds, without any appeal to geometrical intuitions — considerations which may properly be postponed until a later stage in the student's mathematical study.

Turning now to the main part of the book, on algebra proper, we find a full discussion of "everything relating to algebra that

* Cf. *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 209–229, especially § 10.

a student is likely to need during his school and college course." The only topic which one will seek in vain is the binary scale of notation, which the author would regard, I suppose, as belonging more properly in arithmetic. The treatment of the various topics is fresh and stimulating, and is well calculated to keep before the student, in their proper relations, the two essentials of a mastery of the subject, namely sound reasoning and skillful manipulation.

It is of course true that no human reasoning can be more than approximately rigorous, and that in elementary work it is not desirable to carry this approximation beyond the limits of understanding of the pupil; but surely some advance over our current text-books is to be sought. A good example of what seems to me a proper degree of rigor in an elementary text-book is the excellent discussion of the solution of equations on page 114: "It is important to remember that the mere fact that a certain value of x has been derived from an equation by the rules of reckoning does not prove it to be a root; the process must be *reversible* to warrant this conclusion." This is not a mere subtlety inserted for the sake of an appearance of rigor; it is a bit of common sense which is absolutely indispensable even if we wish merely to "get the right answer." For example (page 288), in solving the simple irrational equation $x - 7 - \sqrt{x-5} = 0$, we find that *if* x is a root, *then* x is 9 or 6; but it does not follow that (conversely) *if* x is 9 or 6, *then* x is a root; as a matter of fact, the value 6 is not a root. There is no more reason for supposing that the converse of a true proposition is also true in algebra than there is for making the same mistake in geometry. It is to be regretted that the exercises on page 290 were not so chosen as to emphasize this point.

The early introduction of synthetic division and of the method of undetermined coefficients is to be commended, as is also the use of graphs in connection with equations of the first and second degrees; in this graphical discussion, however, it might have been well to point out the sharp distinction between a *curve* as the picture of a function of one variable and a *surface* as the picture of a function of two variables.

The chapter on simultaneous quadratics, while rather difficult, will well repay a careful study, since it replaces the usual list of special devices by a connected treatment of the subject as a logical whole. It should be noticed, however, that equations like $x^2 + 3y^2 = 31$, $7x^2 - 2y^2 = 10$ (page 324) should

be regarded not as quadratic equations in x and y but as linear equations in x^2 and y^2; and the use of infinite solutions is of doubtful value.

In the chapter on the theory of equations it would have been well to give a few examples of numerical equations with irrational coefficients, since the equations which come up in practice are frequently of this type. Again, in the chapter on logarithms, the awkward double-headed rule for finding the characteristic which is now in common use might well have been replaced by the older and simpler rule which refers to the units' place instead of to the decimal point.

The later chapters of the book are a veritable store-house of accurate and well-selected information concerning infinite series, infinite products, continued fractions, etc. Special attention should be called to the last chapter of all, which contains a brief but admirable discussion of the continuity of functions of one and two variables, including correct definitions of such terms as upper and lower limit and oscillation of a function, a proof of Weierstrass's theorem that a continuous function actually reaches both its upper and its lower limit in any closed interval, and Argand's proof of the fundamental theorem of algebra. This chapter is of course not intended for a first-year class, but will be valuable for reference in the later years of the student's course.

It may be noted in conclusion that the book has an excellent index, and that answers to the exercises will be supplied by the publishers on the request of a teacher.

E. V. HUNTINGTON.

FREUND'S TRANSLATION OF BALL'S HISTORY OF MATHEMATICS.

Histoire des Mathématiques. Par W. W. ROUSE BALL. Edition française revue et augmentée. Traduite sur la troisième édition anglaise par L. FREUND. Tome premier. Paris, A. Hermann, 1906. viii + 422 pp.

IT is an interesting fact that France, which produced in Paul Tannery one of the greatest, if not the greatest historian of mathematics of our time, has not for a hundred years produced a great general history of the subject. Libri was a scholarly

collector of material, and a charming writer, but he did not produce a balanced history. Chasles was an investigator of unusual merit, but he never wrote a general history. Marie, with his twelve volumes based on forty years of study, produced only a mass of inaccuracies. Hoefer's little book, although mentioned as an authority by Ball, is unworthy of any attention from considerations either of style or of scholarship. Boyer's works have no merit, and Rebière's are simply amusing. Indeed France seems to have exhausted her powers in Montucla, whose work even yet is a model in style and is not without value as an authority a century and a half after the first edition appeared.

In view of these facts it is gratifying that the translator can in a measure supply what French scholars fail to produce. Zeuthen has recently been published in Paris, the Encyklopädie (a history in the best sense) is also appearing, and various other contributions to history are finding their way into the French language. It is a matter of congratulation that Ball's "Short Account of the History of Mathematics" should be included in this list. For whatever may be said against the work, the fact remains that it is one of the best arranged and most readable handbooks on the subject ever published. It has its weak features, and these have already been mentioned in various journals as the successive editions of the original have appeared, but it is doubtful if any similar work is more often used. Mr. Ball, for example, gives only about sixteen pages to his chapter on the development of arithmetic in the important period from 1300 to 1637, and less than a dozen pages to the primitive arithmetic, and he rarely mentions the subject elsewhere; he makes no mention of the Nana Ghat inscriptions; he shows familiarity with a relatively small range of historical literature, and he derives very little information from the original sources. We can reasonably pardon him for saying that the Quadrivium of Psellus appeared in 1556, although there was an edition published in Venice in 1532, and several other editions appeared before the given date. We can overlook the fact that he gives V. M. de Kempten's Practique as appearing in 1556 instead of 1550; that (page 142) he leads us to infer that the works of Cassiodorus did not appear until 1729, although the Paris edition of 1598 is well known, not to mention numerous other editions of the Compendium; that he gives 1496 as the date of publication of Sacrobosco's Algorismus, although it

appeared in 1488, and that he gives 1502 instead of 1495 for the first publication of Bradwardin's arithmetic; that he asserts so positively that Campanus was a canon at Paris, and that he tells us that Ahmes was a priest, when he was only a scribe. For a not particularly scientific handbook these and similar errors may, perhaps, be overlooked for the pleasure of having at hand such a readable manual. Mr. Ball has a good eye for the selection of important topics and of interesting facts, and this has enabled him to produce a book well worth translating.

The translator has arranged to produce the work in two volumes, with a larger and more open page than the original, and the first of these volumes is the one now under review. The opportunity which came to him was an excellent one for a scholar. A real student of the history of mathematics would have made use of all of the good features of the original, would have corrected the manifest errors, and would have added bibliographical notes of value and amplified the text where it was necessary. Unfortunately Lieutenant Freund has done nothing of the sort. He has merely made a mediocre translation, adding a few footnotes of little value, and showing that his taste is really in the line of physics by an appendix devoted almost exclusively to this subject and to logarithms. His translation is not free from errors, it is carelessly made even when not otherwise objectionable, and it preserves all of the blemishes of the original.

That the work is carelessly done is seen in such features as the common lack of uniformity in titles. Ball's Mathematical Recreations is sometimes cited in French, and sometimes in English, and with the dates 1899 (page 21), 1898 (page 228), each of which should be 1897 according to the advertisement on the cover. (See also pages 31, 39, 128). Paul Tannery appears sometimes as P. and sometimes as S. P. Tannery, although always in the latter form in the original. On page 192 the title of Taylor's work is given differently in the two notes. On page 190, "De Karlsruhe" should not be in small capitals, nor should "Zwickau" on page 222. The fact that a list of names occupying seven lines on page 49 is omitted, for no apparent reason, renders line eight meaningless: "Parmi tous les mathématiciens dont nous venons de donner les noms." In the matter of proper names the treatment is even less satisfactory than in the original. Althought it is gratifying to see such Greek forms as Antiphon and Heron instead of Antipho

and Hero, the French forms are often used even in cases where the classical ones are the more common. The translator also takes liberties that are quite unwarranted, as when he uses sacro-Bosco for Sacrobosco. If he were to change the form used by Ball it would have been better to adopt a more common one like Sacrobusto or Sacro Bosco.

That the footnotes added by Lieutenant Freund are of little value is seen in the fact that there are only a dozen all told. Of these, four mention articles by Loria, one mentions a work by Sedillot, one devotes seven lines to giving the useless information as to who the Admirable Crichton was (and then misspells his name!), and one says that Charles Kingsley wrote a novel about Hypatia. Of recent literature on the history of mathematics there is nothing save the four references to Loria. Hoefer is retained as an authority, but Braunmühl is unknown; Marie is retained, but there is no mention of Zeuthen or of those who are making the Abhandlungen a storehouse of first-hand material.

That no originality of scholarship is shown by the translator in his work is seen not only in the paucity of his footnotes but in the appended matter. This consists merely of a series of extracts from the works of various writers as follows: 1. A brief note by Chasles on Vieta (page 327); 2. An extended note by Biot on Napier and the invention of logarithms (pages 328–353); 3. A note, chiefly from Bertrand, on Kepler (pages 354–358); 4. A note by Mach on the development of the principles of dynamics (pp. 359–409); 5. A note by Duhem on the origin of statics (pages 410–412).

That Lieutenant Freund has preserved all of the errors and doubtful statements of the original is continually manifest. For example, we do not know, as Ball asserts, that the Mexicans used the abacus (page 132); the figure on page 135 is not a Chinese swan pan but a Japanese saroban; Alkwarizmi was born in Khuwarism, not in Khorassan (page 165); we do not know that Fibonnacci was born in 1175 (page 175), nor that Psellus was born in 1020, nor is there justification for similar positive statements not infrequently made as to dates; Widman did not use the + and − "pour indiquer l'addition et la subtraction," but for a purpose explained later in the book; Recorde did not use the symbol Z to indicate equality of ratios, for, to select one of his examples, he says: "I ſet them one ower the other, and 8 vndermoſt, thus $\frac{3}{8}\overline{\smile}$, with ſuch a crooked

draught of lines. Then doe I ſet the other number which is 16, again ſt 3 at the right ſide of the line, thus $\frac{3}{8}\angle^{16}$" In other words, the lines merely indicate a direction of operations, not any equality of ratios.

In general the mere translation seems satisfactory. It is a question however whether Recorde had in mind any such idea as is found in the interpretation of his "Grounde of Artes" as meaning "Jardin des Artes" (page 221).

That the book abounds in typographical errors is shown by the list given below. This is merely the result of a hurried reading, and is doubtless very incomplete. It shows, however, such an utter lack of care on the part of the translator as to assure us that an opportunity has been missed. Such a list makes it seem almost hopeless to expect a worthier effort in the second volume. Page 21, for 1899 read 1897; page 33, for Ænopides read Œnopides; page 39, for Ritz Patrick read Fitz Patrick; page 49, for Ménœchme read Ménæchme, and so on page 414, and on pages 81, 82, change Menechme; page 53, for Grichen read Griechen; and for modern read modernen; page 55, for Euclides read Eucleides (De Morgan's article), and for studien read Studien (the German capitalization being elsewhere followed); page 56, for Strobacus read Strobaeus; page 81, for Grow read Gow; page 86, for F. Hiller read E. Hiller; page 87, for Chrichton read Crichton; page 90, for phœnomena read phænomena; page 93, for J. S. Greenwood read J. G. Greenwood; page 95, for Αὐτομαια read Αὐτόματα; page 101, for Hache read Hoche; page 102, for Ptolemœus read Ptolemæus, and for 3,1416 read 3,1416; page 103, for Συναγωγη read Συναγωγή; page 112, for μ^{δ} read μ^{o} four times; page 112 for ψ read π twice; page 136, for ἑτεως read ετεος, or ἔτεος if the accents are taken, the inscription referring to the seventh year, not to seven years; page 143, for C. Werner read K. Werner (cf. page 145); page 147, for Malmesburg read Malmesbury; page 154, for Appolonius read Apollonius; page 155, for Batha read Bhata; page 156, for M. Kern read H. Kern; page 161, for Arhya read Arya; page 179, for algebriste read algebrista; page 183, for Medita read Inedita; page 186, for Radolt read Ratdolt; page 190, for Encyclopœdia read Encyclopædia, and drop "Supplement" from note 1; page 263, for Breitsewert read Breitschwert; page 287, for $(n-1^2)^{\varepsilon})$ read $(n-1)^2$; page 296, for $(x-x^3)^2$ read $(x-x^2)^1$; page 298, for Schoeten read Schooten (as on page 313, 315);

page 307, for $p\frac{1}{3}a^{\frac{1}{3}}e$ read $p^{\frac{1}{3}}a^{\frac{1}{3}}e$; page 313, for Dictionnary read Dictionary; page 316, for d read ∂ four times, and three times on page 324; page 321, insert *di* after *Bulletino*, as elsewhere.

DAVID EUGENE SMITH.

SHORTER NOTICES.

Abhandlungen zur Geschichte der mathematischen Wissenschaften. 18. Heft. Inhalt; J. L. HEIBERG, Mathematisches zu Aristotles; C. H. MÜLLER, Studien zur Geschichte der Mathematik insbesondere des mathematischen Unterrichts an der Universität Göttingen im 18. Jahrhundert; R. LINDT, Das Prinzip der virtuellen Geschwindigkeiten. Leipzig, B. G. Teubner, 1904. 196 pp. Price, 6 marks.

FROM the standpoint of the history of pure mathematics the first two parts of this volume of the *Abhandlungen* are of great interest and value. The death of Paul Tannery left no one so well prepared to speak with authority upon a question involving both Greek mathematics and literature as Professor Heiberg. Although primarily a student of the classics, this prolific scholar has so long devoted his attention to the ancient mathematicians that he has become one of the great authorities upon their contributions.

The various histories of mathematics have always recognized the impetus given to mathematics by both Plato and Aristotle, by the former in fixing the foundations, and by the latter with reference to the history and the applications of the science. There is, however, a lack of definite information regarding the mathematical contributions of both of these leaders of philosophic thought. If we try to find exactly what Plato contributed to the advance of mathematics, Cantor, Gow, Zeuthen, and even Tannery give answers that are far from satisfactory. For Aristotle this has also been the case, and hence this concontribution of Professor Heiberg is timely and welcome, especially as it throws much light on the work of Plato as well.

The essay opens with a discussion of the sources of information and then by detailed references to the works of Aristotle it shows his influence upon the subsequent work of the Greeks. In particular the influence of this writer upon the Greek mathematical terminology is shown to be much greater than would be

suspected from a reading of the standard histories. Even more interesting is the study of various propositions of Euclid in relation to the writings of Aristotle, since no serious effort has been made to trace definitely the origin of the individual theorems of the Elements. Professor Heiberg also calls attention to the fact that certain propositions of Aristotle were not used by the Alexandrian master, including the one relating to the exterior angle sum of a polygon.

We are so apt to think of Göttingen as a mathematical center of the nineteenth century alone, owing largely to the influence of Gauss, that the article by Dr. Müller on mathematics in that university in the eighteenth century will dispel what has come to be a common illusion. The essay was undertaken with the approval of Professor Klein, and to him and his associates the author acknowledges his indebtedness.

Dr. Müller first discusses the general historical problem in mathematics and then devotes a chapter to Halle and Göttingen as the universities of rationalism, and to the mathematics of this great educational period. In this chapter he discusses the scientific work of the universities of the sixteenth and seventeenth centuries in general, and of the early days of Halle and Göttingen in particular. Chapter II considers the mathematics of the period of rationalism at Göttingen, and particularly the work of Segner in pure science and of Penther in the applied field. The third chapter is devoted to the mathematics of the period of Enlightenment (Aufklärung) and chiefly to the work of Kästner and Meister. The work closes with the rise of the new humanism at Göttingen, considering particularly the later work of Kästner and his followers.

The early university courses in mathematics, and the general attitude of the academic world towards this science, are among the most interesting features of the essay. Von Rohr's exposition of Wolf's summary, for example, is very suggestive of the position of mathematics in the eighteenth century. For most Americans, the fact that Kästner (1749) thought a work of one of our countrymen * worthy of translation, is a matter of no little surprise and interest.

DAVID EUGENE SMITH.

* An exposition of the first causes of action in matter, and of the cause of gravitation, by Cadwallader Colden, N. Y., 1745.

Elemente der Vector-Analysis. Von A. H. BUCHERER. 2te Auflage, Leipzig, B. G. Teubner, 1905. viii + 103 pp.

SINCE the first edition of this book, two years ago, there have appeared several books on vector analysis and yet a second edition of this one is called for. This is an indication of the rapidly spreading use of vectorial methods in physics and a testimony to the completeness and compactness of Bucherer's little book. As the first edition was recently reviewed in the BULLETIN,* it will merely be necessary to point out the changes in the new edition. The Heaviside-Föppl notation has been abandoned for that of Lorentz and Abraham — an evident effect of the usage of the Encyklopädie. A few sections have been inserted here and there — one on division of vectors, another on the motion of an electron; but generally speaking the increase of twelve pages is due to minor alterations. As the author is very anxious to keep his book within small bounds he still resists the temptation to enter on the study of linear vector functions. So many authors do this same thing that the use of linear functions in the theory of elasticity and light, where they are a great convenience, is seriously hampered.

E. B. WILSON.

Annuaire du Bureau des Longitudes pour l'An 1906. Paris, Gauthier-Villars.

THERE have been no changes in the *Annuaire* this year to call for special remark. The notices are all devoted to eclipse literature, with special reference to the phenomenon which occurred last August. M. G. Bigourdan, however, goes into much detail on the question of the different kinds of observations which can be made during the few minutes duration of a total eclipse of the sun. To professional astronomers this summary will be found useful as containing in a handy form facts and ideas with which they are familiar, and to amateurs it will be of equal value as showing and explaining easily the many observations which it is possible for them to undertake and carry through with success. In a second article the same writer gives a brief account of the numerous expeditions which were made last summer to observe the eclipse. M. Janssen contributes an article on his own share of this work in Spain.

In reading through these articles in the *Annuaire* during the

* Volume 10, February, 1904, pp. 263–267.

last ten years, the reviewer has been frequently struck with the lucidity and ease shown by the writers in explaining even the most technical parts of mathematical and physical problems. Is it impossible to do this in the English language? And if not, why are such summaries so rarely seen? Or, if published, why do they seem to be heavy and unattractive? Surely it cannot be the fault of the language when we have before our eyes such a master of scientific style as Huxley. Perhaps there is something to be learned from France amongst the methods which she uses in teaching her sons to write their mother tongue. ERNEST W. BROWN.

Astronomical and Historical Chronology. By W. L. JORDAN. London, Longmans, Green & Co. 8vo. 70 pp.

THE object of this little book will be sufficiently gathered from the author's statement on page 9: "My argument shows that through a misunderstanding on the part of comparatively modern historians they treated as 1 B. C. the year which, when the era was first established, was called 1 A. D. by those who used ordinal, and the year 0 by those who used cardinal numbers; and that the manner in which the centuries are considered to be divided is therefore erroneous." The question is discussed historically and much space is given to an examination of the authorities. Mr. Jordan comes to the conclusion that if the year 0 be inserted, January 1, 1900 is the beginning of the new century as decreed by the English Prayer Book and the German emperor; but that with the 'vulgar' chronology, which makes 1 B. C. immediately precede 1 A. D., the new century began a year later. The author's physical ideas seem somewhat vague: he alludes (page 33) "to the absence of any common measure between days and years as being due to the fact that the motions which they respectively measure are due to the action of two independent forces — the sun's and the earth's revolving force — etc." But perhaps this is unfair: he has written an essay on the action of astral gravitation in natural phenomena. ERNEST W. BROWN.

NOTES.

AT the meeting of the London mathematical society held on January 11 the following papers were read: By Mr. J. W. NICHOLSON "On the diffraction of sound by large cylinders"; by

Dr. H. F. BAKER, "On the monogeneity of an algebraic function"; by Mr. M. J. BRILL, "On the expression of the so-called biquaternions and triquaternions with the aid of quaternary matrices"; by Dr. E. W. HOBSON, "On the representation of functions of real variables."

THE second meeting of the Missouri society of teachers of mathematics was held at Jefferson City in conjunction with the state teachers' association, December 27 and 28. Two afternoon sessions were held, at which the following papers were presented: "Maxima and minima," by G. R. DEAN; "Laboratory methods in algebra teaching," by O. E. GLENN; "The treatment of limits in elementary geometry," by A. M. WILSON. The second session was devoted to a general discussion of the question: "What should be taught in arithmetic"? Arrangements were made to enlarge the scope of the society so as to include the natural science teachers of the state.

THE association of the teachers of mathematics of the middle schools of Switzerland met at Zürich, December 9, under the presidency of Dr. E. GUBLER. The following papers were read: "Instruction in descriptive geometry," by C. EGLI; "Tendencies in the instruction of elementary geometry," by H. FEHR; "Advantages of the decimal division of angles with logarithms to four places," by M. OTTI. Professor H. FEHR was re-elected president. The next meeting will be held in October, 1906.

THE next meeting of the French association for the advancement of science will be held at Lyons, beginning August 2, under the presidency of Professor M. Lippmann, of the University of Paris.

DURING the year 1905 doctorates in mathematics were conferred by the University of Paris on the following candidates (the title of the dissertation is appended in each case): L. ZORETTI, On uniform analytic functions which possess a perfect discontinuous set of singular points; P. STOENESCO, On the propagation and extinction of plane waves in a homogeneous and translucent medium having a plane of symmetry; D. POMPEIU, On the continuity of functions of a complex variable; R. B. DE M. DE BALLORE, On continuous algebraic fractions; A. HUSSON, Application of algebraic integrals in the movement of a heavy solid around a fixed point; J. REVEILLE, Synthetic and an-

alytic study of the displacement of a system which remains similar to itself; C. MONTEIL, Contribution to the study of currents of heat convection.

THE Paris academy of sciences announces the following prizes in mathematical sciences for 1907 and 1908. Competing memoirs should be in the hands of the secretary before the end of the year preceding that in which the respective prizes are awarded. For further details regarding those to be awarded next December, see BULLETIN, volume 11, page 337. The Poncelet prize of 2000 francs will be awarded in 1908 in pure mathematics; Grand prize of 3000 francs in 1908 for important progress in the study of the deformation of quadric surfaces; Montyon prize of 700 francs (annual) for discoveries and inventions in mechanics; Poncelet prize of 2000 francs in 1907 in applied mathematics; Fourneyron prize of 1000 francs will be awarded in 1908 for the most important advances in the theory and use of the steam turbine; Vaillant prize of 4000 francs in 1908 for perfecting the application of the principles of dynamics of fluids to the theory of the helix; Lalande prize of 540 francs (annual) for general progress in astronomy; Guzman prize of 100,000 francs for communication with any star or any planet other than Mars. As this prize will probably not be soon awarded, the capital has been invested and the accrued interest will be given as a prize every five years for important progress in astronomy. Valz prize of 460 francs (annual) for general progress in astronomy; Pontécoulant prize of 700 francs (biennial) will be awarded in 1907 for progress in celestial mechanics; Damoiseau prize of 2000 francs (triennial) will be awarded in 1908 for the consistent theory of a planet, based upon all known observations; Hébert prize of 1000 francs (annual) for discoveries or advances in the theory of electricity; Hughes prize of 2500 franc (annual) for general physics; Gaston Planté prize of 3000 francs (biennial, open only to Frenchmen) will be awarded in 1907 for electricity; La Caze prize of 10,000 francs (biennial) will be awarded in 1907 for general physics; Kastner Boursalt prize of 2000 francs (biennial) will be awarded in 1907 for progress in the application of electricity; Binoux prize of 2000 francs (1907) for the best contribution to the history of the sciences. Besides the general prizes mentioned before, a new prize known as the Leconte prize of 50,000 francs is to be awarded in 1907 and every

three years thereafter for important discoveries in mathematical or natural sciences; it is open to all competitors.

A MEMORIAL to the late Dr. George Salmon, Provost of Trinity College, Dublin, was unveiled Friday, January 5, in the national cathedral of St. Patricks, Dublin. It consists of two windows in one of the chapels, a portrait of Dr. Salmon, and a Latin inscription bearing testimony to his mathematical and theological work.

PROFESSOR P. STĀCKEL, of the technical school at Hanover, has received the order of the iron cross of the third class from the Emperor of Austria.

PROFESSOR J. THOMAE, of the University of Jena, has been decorated with the order of the cross of Saxe-Weimar by the King of Saxony.

PROFESSOR W. v. DYCK, of the technical school at Munich, has been awarded the Bavarian silver medal by Prince Luitpold.

DR. H. BRUNN has been promoted to an honorary associate professorship of mathematics at the University of Munich.

PROFESSOR L. SAALSCHÜTZ, of the University of Königsberg, celebrated his seventieth birthday December 1, 1905.

DR. G. FUBINI has been promoted to an associate professorship of higher analysis at the University of Catania.

DR. G. BISCONCINI has been appointed docent of rational mechanics at the University of Rome.

PROFESSOR A. FAVARO, of the University of Padua, will henceforth give a regular annual course of lectures on the history of mathematics.

PROFESSOR G. BAGNERA has been promoted to a full professorship of higher analysis at the University of Messina.

PROFESSOR E. BORTOLOTTI, of the University of Modena, has been appointed to a full professorship of the calculus at the same institution.

PROFESSOR SIMON NEWCOMB has been decorated with the Prussian order Pour le Mérite by the German emperor.

PROFESSOR O. S. STETSON has been promoted to an assistant professorship of mathematics in Syracuse University.

MR. GEORGE BRETT has been elected tutor in mathematics in the College of the City of New York.

PROFESSOR C. A. VAN VELZER, head professor of mathematics at the University of Wisconsin, has resigned, the resignation to go into effect July 1, 1906. He expects to retire altogether from teaching.

WE are informed by Professor FLORIAN CAJORI that the John Crerar Library in Chicago, has a complete set of the *Maandelykse Mathematische Liefhebbery*, 1754–1769, published in 17 volumes at Purmerende in the Netherlands. This is probably the oldest journal devoted to *elementary* mathematics. It devotes itself mainly to arithmetic and algebra, and contains several oddities in notation.

RECENT second-hand catalogue: List und Francke, Talstrasse 2, Leipzig, catalogue No. 377, 1,121 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. D'). Trois maîtres: Ampère, Cauchy, Hermite. (Extrait de *La Quinzaine.*) 1905. 8vo. 15 pp.

BIRCHBY (W. N.). Euler's summation of series of reciprocal powers and related series. (*Colorado College Publications*, Vol. 11, pp. 191–208.) Colorado Springs, 1905. 8vo.

BRUNS (H.). Wahrscheinlichkeitsrechnung und Kollektivmasslehre. (B. G. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften.) Leipzig, Teubner, 1906. 8vo. 8 + 310 + 18 pp. Cloth. M. 8.40

CAUCHY (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des sciences et sous les auspices de M. le ministre de l'instruction publique. 2e série. Tome I. Paris, Gauthier-Villars, 1905. 4to. 574 pp. Fr. 25.00

DOEHLEMANN (K.). Projektive Geometrie in synthetischer Behandlung. 3te, vermehrte und verbesserte Auflage. (Sammlung Göschen, 72.) Leipzig, Göschen, 1905. 16mo. 181 pp. Cloth. M. 0.80

GULDBERG (A.). Sur une classification des problèmes du calcul des variations. (*Rendiconti del Circolo Matematico di Palermo*, Tomo XXI.) 8vo. 9 pp.

GUTZMER (A.). See VIVANTI (G.).

LAGRANGE (J. L.). Ueber die Lösung der unbestimmten Probleme 2. Grades. Uebersetzt und herausgegeben von E. Netto. Leipzig, 1904. 8vo. 131 pp. Cloth. M. 2.20

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